Lecture 6 : Derivatives and Rates of Change

In this section we return to the problem of finding the equation of a tangent line to a curve, $y = f(x)$. If $P(a, f(a))$ is a point on the curve $y = f(x)$ and $Q(x, f(x))$ is a point on the curve near P, then the slope of the secant line through P and Q is given by

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$

Our intuition tells us that the slope of the tangent line to the curve at the point P is

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to a} m_{PQ} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

With this in mind, we make the following definition:

Definition When $f(x)$ is defined in an open interval containing a, the **Tangent Line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

Example Find the equation of the tangent line to the curve $y = \sqrt{x}$ at $P(1, 1)$.
(Note: This is the problem we solved in Lecture 2 by calculating the limit of the slopes of the secants. At this point we have become more efficient with our calculations of limits.)

$m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} =$

The equation of the tangent to the curve at $P(1, 1)$ is :

Note The limit in the definition above can be rewritten as follows:

$$m \quad = \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad = \quad \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

This limit gives the slope of the line tangent to the curve $y = f(x)$ at $P(a, f(a))$ if the limit exists. The slope of the tangent line to a curve at a point (when it exists) is sometimes called the slope of the curve at that point. y-values on the curve "near the point" are close to corresponding y-values on the tangent line. (We will examine this property more closely when we get to Linear Approximation).

Example Find the equation of the tangent line to the graph of $f(x) = x^2 + 5x$ at the point $(1, 6)$.

$m = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} =$

Derivative of a function at a point a

Definition When $f(x)$ is defined in an open interval containing a, the **derivative** of the function f at the number a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exists.

Note The slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$ is the derivative of f at a, $f'(a)$.

Example Let $f(x) = x^2 + 5x$. Find $f'(a)$, $f'(2)$ and $f'(-1)$.

Equation of the Tangent Line Note that the equation of the tangent line to the graph of a function f at the point $(a, f(a))$ is given by

$$(y - f(a)) = f'(a)(x - a).$$

Example Find the equation of the tangent line to the graph $y = x^2 + 5x$ at the point where $x = 2$.

Find the equation of the tangent line to the graph $y = x^2 + 5x$ at the point where $x = -1$.

Note When the derivative of a function f at a, is positive, the function is increasing and when it is negative, the function is decreasing. When the absolute value of the derivative is small, the function is changing slowly (a small change in the value of x leads to a small change in the value of $f(x)$). When the absolute value of the derivative is large, the function values are changing rapidly (a small change in x leads to a large change in $f(x)$).

Some limits are easy to calculate when we recognize them as derivatives:

Example The following limits represent the derivative of a function f at a number a. In each case, what is $f(x)$ and a?

$$(a)\quad \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \frac{1}{\sqrt{2}}}{x - \pi/4} \qquad\qquad (b)\quad \lim_{h\to 0} \frac{(1+h)^4 + (1+h) - 2}{h}$$

(a) $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \frac{1}{\sqrt{2}}}{x-\pi/4} = \lim_{x\to\frac{\pi}{4}} \frac{f(x)-f(a)}{x-\pi/4}$.
f(x) =
a =

(b) $\lim_{h\to 0} \frac{(1+h)^4+(1+h)-2}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.
f(x) =
a =

Velocity

If an object moves in a straight line, the displacement from the origin at time t is given by the **position function** $s = f(t)$, where s is the displacement of the object from the origin at time t. The **average velocity** of the object over the time interval $[t_1, t_2]$ is given by

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

The **velocity (or instantaneous velocity)** at time $t = a$ is given by the following limit of average velocities:

$$v(a) = \lim_{t\to a} \frac{f(t) - f(a)}{t - a} \quad = \quad \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

Thus the velocity at time $t = a$ is the slope of the tangent line to the curve $y = s = f(t)$ at the point where $t = a$.

Example The position function of a stone thrown from a bridge is given by $s(t) = 10t - 16t^2$ feet (below the bridge) after t seconds.

(a) What is the average velocity of the stone between $t_1 = 1$ and $t_2 = 5$ seconds?

(b) What is the instantaneous velocity of the stone at $t = 1$ second. (Note that speed = |Velocity|).

Different Notation, Rates of change, Δx, Δy

If y is a function of x, $y = f(x)$, a change in x from x_1 to x_2 is sometimes denoted by $\Delta x = x_2 - x_1$ and the corresponding change in y is denoted by $\Delta y = f(x_2) - f(x_1)$. The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x**. This is the slope of the line segment PQ, where $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ are on the graph $y = f(x)$.

The **instantaneous rate of change of y with respect to x**, when $x = x_1$, is the limit of the slopes of line segments PQ as Q gets closer and closer to P on the graph $y = f(x)$;

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad (= f'(x_1)).$$

Note that this is just the derivative of $f(x)$ when $x = x_1$. Thus we have another interpretation of the derivative:

The derivative, $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$. When the instantaneous rate of change is large at x_1, the y-vlaues on the curve are changing rapidly and the tangent has a large slope. When the instantaneous rate of change ssmall at x_1, the y-vlaues on the curve are changing slowly and the tangent has a small slope.

In economics, the instantaneous rate of change of the cost function (revenue function) is called the **Marginal Cost** (Marginal Revenue).

Example The cost (in dollars) of producing x units of a certain commodity is $C(x) = 50 + \sqrt{x}$.

(a) Find the average rate of change of C with respect to x when the production level is changed from $x = 100$ to $x = 169$.

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$ (Marginal cost when $x = 100$, usually explained as the cost of producing an extra unit when your production level is 100).

Example The cost (in dollars) of producing x units of a certain commodity is $C(x) = 50 + \sqrt{x}$.

(a) Find the average rate of change of C with respect to x when the production level is changed from $x = 100$ to $x = 169$.

Solution The average rate of change of C is the average cost per unit when we increase production from $x_1 = 100$ tp $x_2 = 169$ units. It is given by

$$\frac{\Delta x}{\Delta y} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{50 + \sqrt{169} - (50 + \sqrt{100})}{169 - 100} = \frac{13 - 10}{69} = \frac{3}{69} = .04347.$$

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$ (Marginal cost when $x = 100$, usually explained as the cost of producing an extra unit when your production level is 100).

Solution The instantaneous rate of change of C when $x = 100$ It is given by

$$\lim_{x \to 100} \frac{\Delta x}{\Delta y} = \lim_{x \to 100} \frac{f(x) - f(100)}{x - 100} = \lim_{x \to 100} \frac{50 + \sqrt{x} - (50 + \sqrt{100})}{x - 100} = \lim_{x \to 100} \frac{\sqrt{x} - 10}{x - 100}$$

$$= \lim_{x \to 100} \frac{(\sqrt{x} - 10)}{(\sqrt{x} - 10)(\sqrt{x} + 10)} == \lim_{x \to 100} \frac{1}{(\sqrt{x} + 10)} = \frac{1}{20} = .05$$